

Classification of Good Gradings of Simple Lie Algebras

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ABSTRACT. We study and give a complete classification of good \mathbb{Z} -gradings of all simple finite-dimensional Lie algebras. This problem arose in the quantum Hamiltonian reduction for affine Lie algebras.

0. Introduction

Let \mathfrak{g} be a finite-dimensional Lie algebra over an algebraically closed field \mathbb{F} of characteristic 0. Let $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ be a \mathbb{Z} -grading of \mathfrak{g} (i.e., $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$), and let $\mathfrak{g}_+ = \bigoplus_{j > 0} \mathfrak{g}_j$, $\mathfrak{g}_{\geq} = \bigoplus_{j \geq 0} \mathfrak{g}_j$.

An element $e \in \mathfrak{g}_2$ is called *good* if the following properties hold:

$$(0.1) \quad \text{ad } e : \mathfrak{g}_j \rightarrow \mathfrak{g}_{j+2} \text{ is injective for } j \leq -1,$$

$$(0.2) \quad \text{ad } e : \mathfrak{g}_j \rightarrow \mathfrak{g}_{j+2} \text{ is surjective for } j \geq -1.$$

Obviously, e is a non-zero nilpotent element of \mathfrak{g} . Note that (0.1) is equivalent to

$$(0.3) \quad \text{the centralizer } \mathfrak{g}^e \text{ of } e \text{ lies in } \mathfrak{g}_{\geq}.$$

Also, (0.1) and (0.2) for $j = -1$ imply that

$$(0.4) \quad \text{ad } e : \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_1 \text{ is bijective.}$$

Finally, (0.2) for $j = 0$ means that

$$(0.5) \quad [\mathfrak{g}_0, e] = \mathfrak{g}_2.$$

Denote by G the adjoint group corresponding to the Lie algebra \mathfrak{g} and by G_0 its subgroup consisting of the elements preserving the \mathbb{Z} -grading. Then (0.5) implies that $G_0 \cdot e$ is a Zariski dense open orbit in \mathfrak{g}_2 . Consequently, all good elements form a single G_0 -orbit in \mathfrak{g}_2 , which is Zariski open.

A \mathbb{Z} -grading of \mathfrak{g} is called *good* if it admits a good element.

Supported in part by SFB/TR 12 at the Ruhr-University Bochum.

Supported in part by NSF grant DMS-0201017.

The most important examples of good \mathbb{Z} -gradings of \mathfrak{g} correspond to sl_2 -triples $\{e, h, f\}$, where $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. It follows from representation theory of sl_2 that the eigenspace decomposition of $\text{ad } h$ in \mathfrak{g} is a \mathbb{Z} -grading of \mathfrak{g} with a good element e . We call the good \mathbb{Z} -gradings thus obtained the *Dynkin \mathbb{Z} -gradings*.

In the present paper we classify all good \mathbb{Z} -gradings of simple Lie algebras. More precisely, for each nilpotent element e of a simple Lie algebra \mathfrak{g} , we find all good \mathbb{Z} -gradings of \mathfrak{g} for which e is a good element. This problem arose in the study of a family of vertex algebras obtained from an affine (super)algebra, associated to a simple finite-dimensional Lie (super)algebra \mathfrak{g} with a non-degenerate invariant bilinear form, by the quantum Hamiltonian reduction (cf. [FORTW], [KRW], [KW]). The method developed in the present paper works in the “super” case as well. We would like also to point out that the notion of a good grading helps to prove new results even for the well studied case of Dynkin gradings, like Theorem 1.5.

The first named author is grateful to V. Ginzburg for the reference [FORTW], to D. Panyushev for useful discussions, and especially to M. Jibladze, the author of Proposition 4.1, for being a patient listener. We are grateful to K. Baur and N. Wallach for pointing out in math.RT/0409295 an omission in Theorem 6.4.

1. Properties of good gradings

From now on, we shall assume that \mathfrak{g} is a semisimple Lie algebra. Fix a \mathbb{Z} -grading of \mathfrak{g} :

$$(1.1) \quad \mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j.$$

LEMMA 1.1. *Let $e \in \mathfrak{g}_2$, $e \neq 0$. Then there exists $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-1}$ such that $\{e, h, f\}$ form an sl_2 -triple, i.e., $[h, e] = 2e$, $[e, f] = h$, $[h, f] = -2f$.*

PROOF. By the Jacobson–Morozov theorem (see [J]), there exist $h, f \in \mathfrak{g}$ such that $\{e, h, f\}$ is an sl_2 -triple. We write $h = \sum_{j \in \mathbb{Z}} h_j$, $f = \sum_{j \in \mathbb{Z}} f_j$ according to the given \mathbb{Z} -grading of \mathfrak{g} . Then $[h_0, e] = 2e$ and $[e, \mathfrak{g}] \ni h_0$ (since $[e, f_{-2}] = h_0$). Therefore, by Morozov’s lemma (see [J]), there exists f' such that $\{e, h, f'\}$ is an sl_2 -triple. But then $\{e, h_0, f'_0\}$ is an sl_2 -triple. \square

The following lemma is well-known [C] (and easy to prove).

LEMMA 1.2. *Let e be a non-zero nilpotent element of \mathfrak{g} and let $\mathfrak{s} = \{e, h, f\}$ be an sl_2 -triple. Then $\mathfrak{g}^{\mathfrak{s}}$ (the centralizer of \mathfrak{s} in \mathfrak{g}) is a maximal reductive subalgebra of \mathfrak{g}^e .*

THEOREM 1.1. *Let (1.1) be a good \mathbb{Z} -grading and $e \in \mathfrak{g}_2$ a good element. Let $H \in \mathfrak{g}$ be the element defining the \mathbb{Z} -grading (i.e., $\mathfrak{g}_j = \{a \in \mathfrak{g} | [H, a] = ja\}$), and let $\mathfrak{s} = \{e, h, f\}$ be an sl_2 -triple given by Lemma 1.1. Then $z := H - h$ lies in the center of $\mathfrak{g}^{\mathfrak{s}}$.*

PROOF. The element H exists since all derivations of \mathfrak{g} are inner. By (0.3) the eigenvalues of $\text{ad } H$ on \mathfrak{g}^e are non-negative. Hence the eigenvalues of $\text{ad } H$ on \mathfrak{g}^s are non-negative. Since by Lemma 1.2, \mathfrak{g}^s is reductive, it follows that $[H, \mathfrak{g}^s] = 0$. But, obviously, $[h, \mathfrak{g}^s] = 0$, hence $[z, \mathfrak{g}^s] = 0$. \square

COROLLARY 1.1. *If $\mathfrak{s} = \{e, h, f\}$ is an sl_2 -triple in \mathfrak{g} and the center of \mathfrak{g}^s is trivial, then the only good grading for which e is a good element is the Dynkin grading.*

It is well known that \mathfrak{g}_0 is a reductive subalgebra of \mathfrak{g} and a Cartan subalgebra \mathfrak{h} of \mathfrak{g}_0 is a Cartan subalgebra of \mathfrak{g} . Let $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha} \mathfrak{g}_{\alpha})$ be the root space decomposition of \mathfrak{g} with respect to \mathfrak{h} . Let Δ_0^+ be a system of positive roots of the subalgebra \mathfrak{g}_0 . It is well known that $\Delta^+ = \Delta_0^+ \cup \{\alpha \mid \mathfrak{g}_{\alpha} \subset \mathfrak{g}_s, s > 0\}$ is a set of positive roots of \mathfrak{g} . Let $\Pi = \{\alpha_1, \dots, \alpha_r\} \subset \Delta^+$ be the set of simple roots. Setting $\Pi_s = \{\alpha \in \Pi \mid \mathfrak{g}_{\alpha} \subset \mathfrak{g}_s\}$ we obtain a decomposition of Π into a disjoint union of subsets $\Pi = \bigcup_{s \geq 0} \Pi_s$. This decomposition is called the *characteristic* of the \mathbb{Z} -grading (1.1). So we obtain a bijection between all \mathbb{Z} -gradings of \mathfrak{g} up to conjugation and all characteristics. A \mathbb{Z} -grading is called *even* if its characteristic is $\Pi = \Pi_0 \cup \Pi_2$.

THEOREM 1.2. *If the \mathbb{Z} -grading (1.1) is good, then $\Pi = \Pi_0 \cup \Pi_1 \cup \Pi_2$.*

PROOF. Let $e \in \mathfrak{g}_2$ be a good element, and suppose that $\alpha_j \notin \Pi_0 \cup \Pi_1 \cup \Pi_2$ for some j . Then e lies in the subalgebra generated by the e_{α_i} , $i \neq j$, and therefore $[e, e_{-\alpha_j}] = 0$. This contradicts (0.1). \square

This result was proved by Dynkin (see [D]) for the Dynkin gradings.

COROLLARY 1.2. *All good \mathbb{Z} -gradings are among those defined by $\deg e_{\alpha_i} = -\deg e_{-\alpha_i} = 0, 1$ or 2 , $i = 1, \dots, r$.*

THEOREM 1.3. *Properties (0.1) and (0.2) of a \mathbb{Z} -grading $\mathfrak{g} = \bigoplus_j \mathfrak{g}_j$ are equivalent.*

PROOF. The property $[e, \mathfrak{g}_j] \neq \mathfrak{g}_{j+2}$ for $j \geq -1$ is equivalent to the existence of a non-zero $a \in \mathfrak{g}_{-j-2}$ such that $([e, \mathfrak{g}_j], a) = 0$, where (\cdot, \cdot) is a non-degenerate invariant bilinear form on \mathfrak{g} . But the latter equality is equivalent to $([e, a], \mathfrak{g}_j) = 0$, which is equivalent to $a \in \mathfrak{g}_{-j-2}^e$. i.e., to $\text{ad } e$ being not injective on \mathfrak{g}_{-j-2} . \square

THEOREM 1.4. *Let $\mathfrak{g} = \bigoplus_j \mathfrak{g}_j$ be a good \mathbb{Z} -grading with a good element e . Then $\mathfrak{g}^e \cong \mathfrak{g}_0 + \mathfrak{g}_{-1}$ as \mathfrak{g}_0^e -modules.*

PROOF. Due to (0.1) and (0.2) we have the following exact sequence of \mathfrak{g}_0^e -modules:

$$0 \rightarrow \mathfrak{g}^e \rightarrow \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_+ \xrightarrow{\text{ad } e} \mathfrak{g}_+ \rightarrow 0.$$

\square

COROLLARY 1.3. *Let $\mathfrak{g} = \bigoplus_j \mathfrak{g}_j$ be a \mathbb{Z} -grading and let $e \in \mathfrak{g}_2$. Then $\dim \mathfrak{g}^e \geq \dim \mathfrak{g}_{-1} + \dim \mathfrak{g}_0$, and equality holds iff e is a good element.*

PROOF. We have an exact sequence of vector spaces:

$$0 \rightarrow \mathfrak{g}^e \cap (\mathfrak{g}_{-1} + \mathfrak{g}_{\geq 0}) \rightarrow \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_+ \xrightarrow{\text{ad } e} [e, \mathfrak{g}_{-1} + \mathfrak{g}_{\geq 0}] \rightarrow 0.$$

Hence $\dim \mathfrak{g}^e + \dim [e, \mathfrak{g}_{-1} + \mathfrak{g}_{\geq 0}] \geq \dim \mathfrak{g}_{-1} + \dim \mathfrak{g}_0 + \dim \mathfrak{g}_+$. Since $\dim [e, \mathfrak{g}_{-1} + \mathfrak{g}_{\geq 0}] \leq \dim \mathfrak{g}_+$ and the equality holds only if e is good, the corollary follows. \square

This result was proved in [FORTW] for even gradings.

COROLLARY 1.4. *Under the conditions of Theorem 1.4, the representation of \mathfrak{g}_0^e on \mathfrak{g}^e is self-dual (i.e., it is equivalent to its dual).*

PROOF. Consider the bilinear form $\langle a, b \rangle = (e, [a, b])$ on \mathfrak{g}_{-1} . It is \mathfrak{g}_0^e -invariant and, by (0.4), it is non-degenerate. Hence the \mathfrak{g}_0^e -module \mathfrak{g}_{-1} is self-dual. The \mathfrak{g}_0^e -module \mathfrak{g}_0 is self-dual since the bilinear form (\cdot, \cdot) is non-degenerate on \mathfrak{g}_0 . \square

THEOREM 1.5. *Let $\mathfrak{g} = \bigoplus_j \mathfrak{g}_j$ be a good \mathbb{Z} -grading with a good element e . Let \mathfrak{t} be a maximal ad-diagonalizable subalgebra of $\mathfrak{g}^e \cap \mathfrak{g}_0$. Then all the weights of \mathfrak{t} on \mathfrak{g}_1 are non-zero.*

PROOF. Let Z be the centralizer of \mathfrak{t} and let $Z' = Z/\mathfrak{t}$. The \mathbb{Z} -grading of \mathfrak{g} induces a good \mathbb{Z} -grading $Z' = \bigoplus_j Z'_j$ with a good element e' , which is the canonical image of e . Note that the centralizer of e in Z'_0 consists of nilpotent elements, hence by the graded Engel theorem [J], the centralizer of e' in Z' consists of nilpotent elements. Hence, by Corollary 1.1, the \mathbb{Z} -grading of Z' is a Dynkin grading. But it is known that if the centralizer of a nilpotent element e' in a reductive Lie algebra Z' consists of nilpotent elements, then, for the corresponding Dynkin grading, $Z'_1 = 0$ [C]. This proves the theorem. \square

2. Some examples

The most popular conjugacy classes of nilpotent elements in a simple Lie algebra \mathfrak{g} are the following three: (a) the unique nilpotent orbit of codimension $r (= \text{rank } \mathfrak{g})$, called the regular nilpotent orbit, (b) the unique nilpotent orbit of codimension $r + 2$, called the subregular nilpotent orbit, (c) the unique (non-zero) nilpotent orbit of minimal dimension, called the minimal nilpotent orbit.

The characteristic of a regular nilpotent element e is $\Pi = \Pi_2$. The reductive part of its centralizer is trivial, hence by Corollary 1.1, the only \mathbb{Z} -grading for which e is a good element is the Dynkin grading.

Any simple Lie algebra \mathfrak{g} of rank r has r even \mathbb{Z} -gradings for which $\dim \mathfrak{g}_0 = r + 2$, corresponding to the characteristics $(j = 1, \dots, r)$: $\Pi_0 = \Pi \setminus \{\alpha_j\}$, $\Pi_2 = \{\alpha_j\}$. If e is a good element for such a \mathbb{Z} -grading, then by Corollary 1.3, e is a subregular nilpotent element. Furthermore, if \mathfrak{g} is of

type different from A_r or B_r , then the reductive part of \mathfrak{g}^e is trivial, hence by Corollary 1.1, only one of the above r even \mathbb{Z} -gradings of \mathfrak{g} is good, and it is the Dynkin grading. We will show that for the types A_r and B_r , all of the above r even \mathbb{Z} -gradings are good.

The reductive part of the centralizer of a minimal nilpotent element e is semisimple for all types except for A_r . Hence, if \mathfrak{g} is of type different from A_r , e is good only for the Dynkin \mathbb{Z} -grading. We will see that for type A_r all good non-Dynkin \mathbb{Z} -grading with a good minimal nilpotent element have the following characteristics: $\Pi_2 = \{\alpha_1\}$ or $\Pi_2 = \{\alpha_r\}$, $\Pi_0 = \Pi \setminus \Pi_2$, whereas the characteristic of the Dynkin grading is: $\Pi_1 = \{\alpha_1, \alpha_r\}$, $\Pi_0 = \Pi \setminus \Pi_1$.

Recall that \mathfrak{g}_{\geq} is a parabolic subalgebra of \mathfrak{g} and \mathfrak{g}_+ is the nilradical of \mathfrak{g}_{\geq} . An element e from the nilradical of the Lie algebra \mathfrak{p} of a parabolic subgroup $P \subset G$ is called a *Richardson element* for \mathfrak{p} if the orbit Pe is open dense in the nilradical of \mathfrak{p} .

Note that we have an obvious bijective correspondence between even \mathbb{Z} -gradings of \mathfrak{g} and parabolic subalgebras of \mathfrak{g} (by taking \mathfrak{g}_{\geq}).

THEOREM 2.1. *Let $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_{2j}$ be an even \mathbb{Z} -grading, and let \mathfrak{g}_{\geq} be the corresponding parabolic subalgebra of \mathfrak{g} . Then $e \in \mathfrak{g}_2$ is a Richardson element for \mathfrak{g}_{\geq} iff e is good.*

PROOF. It is well known that $\mathfrak{g}^e \subset \mathfrak{p}$ for a Richardson element e of the parabolic subalgebra \mathfrak{p} . Hence if a Richardson element e lies in the \mathfrak{g}_2 then the \mathbb{Z} -grading is good.

Conversely, if e is good, then, by Theorem 1.4, $\dim \mathfrak{g}^e = \dim \mathfrak{g}_0$. Hence, since $\mathfrak{g}^e \subset \mathfrak{g}_{\geq}$, we obtain that $\dim \mathfrak{g}_{\geq}(e) = \dim \mathfrak{g}_{\geq} - \dim \mathfrak{g}_0 = \dim \mathfrak{g}_+$. Hence e is a Richardson element of the parabolic subalgebra \mathfrak{g}_{\geq} . \square

A Richardson element defined by Theorem 2.1 is called *good*.

3. Nilpotent orbits and parabolic subalgebras in classical Lie algebras

Let V be a vector space over \mathbb{F} of finite dimension n . Denote by Gl_n the group of automorphisms of V and by \mathfrak{gl}_n its Lie algebra.

Let $\mathrm{Par}(n)$ be the set of *partitions of n* , i. e. the set of all tuples $p = (p_1, \dots, p_s)$ with $p_i \in \mathbb{N}$, $p_i \geq p_{i+1}$ and $p_1 + \dots + p_s = n$. It will be convenient to assume that in fact p has arbitrary number of further components on the right, all equal to zero, i. e. $p_{s+1} = p_{s+2} = \dots = 0$. We will denote by $\mathrm{mult}_p(j)$ multiplicity of the number j in the partition p , i. e.

$$\mathrm{mult}_p(j) = \#\{i : p_i = j\},$$

so that the partition p can be also written as

$$(3.1) \quad \sum_{i \geq 1} i \mathrm{mult}_p(i) = n.$$

For a partition $p = (p_1, \dots, p_s) \in \text{Par}(n)$ or, more generally, for any sequence (not necessarily decreasing, but such that only finitely many of its entries are positive) we denote by p^* the dual partition, so $p^* = (p_1^*, p_2^*, \dots)$ with $p_j^* := \#\{i : p_i \geq j\}$, $j = 1, 2, \dots$. Note in particular that

$$(3.2) \quad \text{mult}_p(j) = p_j^* - p_{j+1}^*.$$

The nilpotent conjugacy classes in \mathfrak{gl}_n correspond bijectively to the partitions of n (Jordan normal form). So we denote a conjugacy class in Gl_n which corresponds to the partition p , by $\text{Gl}_n(p)$. The following is well known (see [C]):

$$(3.3) \quad \dim \text{Gl}_n(p) = n^2 - ((p_1^*)^2 + (p_2^*)^2 + \dots).$$

The simple Lie algebra \mathfrak{sl}_n (of type A_{n-1}) is the subalgebra of \mathfrak{gl}_n consisting of traceless $n \times n$ matrices over \mathbb{F} . We will take as its Cartan subalgebra \mathfrak{h} its subspace of traceless diagonal matrices. The roots and weights live in the dual \mathfrak{h}^* of \mathfrak{h} , which can be identified with the subspace $x_1 + \dots + x_n = 0$ of \mathbb{R}^n . The roots are $\{e_i - e_j | 1 \leq i \neq j \leq n\}$, and we will choose the positive ones to be $\Delta_+ = \{e_i - e_j | 1 \leq i < j \leq n\}$. The simple roots are then $\alpha_i = e_i - e_{i+1}$, for $1 \leq i \leq n-1$. It is obvious that the description of nilpotent elements, parabolic subalgebras, as well as good gradings, etc, for \mathfrak{gl}_n is the same as for \mathfrak{sl}_n .

It is well known that there exists a bijection between the conjugacy classes of parabolic subalgebras of \mathfrak{gl}_n (and \mathfrak{sl}_n) and compositions of n . An ordered sequence of positive integers (a_1, \dots, a_m) is called a composition of $n = \sum a_i$. Under the above bijection, a parabolic subalgebra corresponding to the composition (a_1, \dots, a_m) is the one consisting of elements preserving a flag $\{0\} = V_0 \subset V_1 \subset \dots \subset V_{m-1} \subset V_m = V$, where $\dim V_i/V_{i-1} = a_i$, $i = 1, \dots, m$.

Consider a $2n$ - (resp. $2n+1$)-dimensional vector space V over \mathbb{F} with a basis $v_1, \dots, v_n, v_{-1}, \dots, v_{-n}$ (resp. $v_1, \dots, v_n, v_0, v_{-1}, \dots, v_{-n}$). In the case of $\dim V = 2n$, denote by $\langle \cdot, \cdot \rangle$ the skew-symmetric bilinear form on V defined by $\langle v_i, v_{-j} \rangle = \delta_{ij}$ for $i \geq 1$. Denote by (\cdot, \cdot) the symmetric bilinear form on V defined by $(v_i, v_{-j}) = \delta_{ij}$. Let $G = Sp_{2n} = \{g \in GL_{2n} | \langle gu, gv \rangle = \langle u, v \rangle, u, v \in V\}$ be the corresponding symplectic group and $\mathfrak{g} = \mathfrak{sp}_{2n}$ its Lie algebra, and let $G = O_N = \{g \in GL_N | (gu, gv) = (u, v), u, v \in V\}$, where $N = 2n$ or $2n+1$, be the corresponding orthogonal group and $\mathfrak{g} = \mathfrak{so}_N$ its Lie algebra.

The parabolic subalgebras of $\mathfrak{g} = \mathfrak{sp}_N$ or \mathfrak{so}_N are described as stabilizers of *isotropic* flags in V , i.e., the flags $V_0 = \{0\} \subset V_1 \subset \dots \subset V_m = V$ such that $V_j^\perp = V_{m-j}$ for $j = 0, 1, \dots, m$. We let $a_i = \dim V_i/V_{i-1}$, $i \geq 1$, let $t = \lfloor m/2 \rfloor$ and $q = 0$ if m is even, and $= a_{t+1}$ if m is odd. Then (a_1, \dots, a_t) is a composition of $\frac{1}{2}(N-q)$. Thus we get a bijective correspondence between conjugacy classes of parabolic subalgebras of \mathfrak{g} and the pairs $(a_1, \dots, a_t; q)$, where q is an integer between 0 and N such that $N-q$ is

even, and (a_1, \dots, a_t) is a composition of $\frac{1}{2}(N - q)$; also, in the case \mathfrak{so}_{2n} , $q \neq 2$.

Denote by $SPar(N)$ (resp. $OPar(N)$) the subset of $Par(N)$ consisting of partitions whose odd (resp. even) parts occur with even multiplicity. It is well known that the nilpotent orbits of G in $\mathfrak{g} = \mathfrak{sp}_N$ and \mathfrak{so}_N correspond bijectively to the partitions from $SPar(N)$ and $OPar(N)$ respectively (see [C]).

A description of the reductive parts of centralizers \mathfrak{g}^e for nilpotent elements e in Lie algebras of classical type can be found in [C]. From this description the following theorem easily follows.

THEOREM 3.1. *Let \mathfrak{g} be a simple Lie algebra of classical type, let $e = e(p)$ be the nilpotent element corresponding to a partition p , and let $c(e)$ be the dimension of the center of \mathfrak{g}^e , the reductive part of the centralizer \mathfrak{g}^e (see Lemma 1.2). Then*

- a) $c(p) = \#(\text{different parts of the partition } p) - 1$ in the case $\mathfrak{g} = \mathfrak{sl}_n$;
- b) $c(p) = \#(\text{odd parts of } p \text{ with multiplicity } 2)$ in the case $\mathfrak{g} = \mathfrak{so}_n$;
- c) $c(p) = \#(\text{even parts of } p \text{ with multiplicity } 2)$ in the case $\mathfrak{g} = \mathfrak{sp}_n$.

4. Good gradings of \mathfrak{gl}_n

A *pyramid* P is a finite collection of boxes of size 1×1 on the plane, centered at (i, j) , where $i, j \in \mathbb{Z}$, such that the second coordinates of the centers of boxes of the lowest row equal 1, the first coordinates of j^{th} row form an arithmetic progression $f_j, f_j + 2, \dots, l_j$ with difference 2, $f_1 = -l_1$, and one has

$$(4.1) \quad f_j \leq f_{j+1}, l_j \geq l_{j+1} \quad \text{for all } j.$$

The *size* of a pyramid is the number of boxes in it.

To a given pyramid P of size n we associate a nilpotent endomorphism of the vector space \mathbb{F}^n in the following manner: enumerate the squares of P in some order, label the standard basis vectors of \mathbb{F}^n by the boxes with the corresponding number, and define an endomorphism $e(P)$ of \mathbb{F}^n by letting it act “along the rows of the pyramid”, i. e., by sending the basis vector labeled by a box to the basis vector labeled by its neighbor on the right, if this neighbor belongs to P and to 0 otherwise. Denote by p_j the number of squares on the j -th row of P . Then the endomorphism $e(P)$ is a nilpotent one corresponding to the partition (i. e. with sizes of Jordan blocks given by) $p = (p_1, \dots, p_k)$ and the endomorphisms corresponding to all pyramids with n boxes and fixed lengths of rows belong to the conjugacy class of the nilpotent $e(P)$. Define a diagonal matrix $h(P) \in \mathfrak{gl}_n$ by letting its j^{th} diagonal entry equal to the first coordinate of the center of the j^{th} box. Then the eigenspace decomposition of $ad(h(P))$ is a \mathbb{Z} -grading of \mathfrak{gl}_n . The characteristic of this \mathbb{Z} -grading can be described as follows. First, denote by h_j the number of squares in the j -th column of P , $j = 1, \dots, 2p_j - 1$, i. e.

such that the first coordinate of their centers is equal to j . Note that for j odd, necessarily $h_j > 0$. Next, for each h_j construct a sequence which begins with $h_j - 1$ zeros and is followed by either 2 — if the right neighbor of h_j is zero; or by 1 — if the right neighbor of h_j is nonzero; or by nothing at all, if h_j does not have any right neighbors (i. e. $j = 2p_1 - 1$). Then concatenate the sequences obtained, to form the sequence of $n - 1$ integers equal 0, 1 or 2, which defines the characteristic in question by assigning these integers to the corresponding simple roots. It is also easy to see that an elementary matrix $E_{i,j}$ has a non-negative degree in this grading iff the label i is not located strictly to the left of the label j in the pyramid P .

Given a partition $p = (p_1, \dots, p_k) \in \text{Par}(n)$, denote by $P(p)$ the symmetric pyramid corresponding to p , i.e. the pyramid with k rows such that the j^{th} row contains p_j boxes centered at (i, j) , where i runs over the arithmetic progression with difference 2 and $f_j = -p_j + 1 = -l_j$. Denote by $\text{Pyr}(p)$ the set of all pyramids attached to the partition p , i.e. the pyramids containing p_j boxes in the j^{th} row, $j = 1, \dots, k$. Obviously, all the pyramids from $\text{Pyr}(p)$ are obtained from the symmetric one, $P(p)$, by a horizontal shift for each $j > 1$ of the boxes of the j^{th} row (as a whole) in such a way that condition (4.1) is satisfied. Hence we obtain the following lemma.

LEMMA 4.1.

$$\# \text{Pyr}(p) = \prod_{i=1}^{k-1} (2(p_i - p_{i+1}) + 1)$$

(here in the case $k = 1$, by the empty product is understood to be 1, as usual).

Using this lemma, we can calculate the generating function

$$F(q) = \sum_n \text{Pyr}_n q^n$$

for the numbers Pyr_n of pyramids of size n . Indeed, using notation from the lemma we obviously have

$$\text{Pyr}_n = \sum_{p \in \text{Par}(n)} \# \text{Pyr}(p).$$

Thus according to the lemma, we can write

$$F(q) = \sum_p \left(\prod_{i: p_{i+1} > 0} (2(p_i - p_{i+1}) + 1) \right) q^{\sum_i p_i},$$

with the sum ranging over all partitions of all natural numbers. Observe now that since partitions are in one-to-one correspondence with duals of partitions, we obviously have

$$F(q) = \sum_p \left(\prod_{i: p_{i+1}^* > 0} (2(p_i^* - p_{i+1}^*) + 1) \right) q^{\sum_i p_i}.$$

Then using 3.1 and 3.2 we can write

$$F(q) = \sum_p \left(\prod_{i: p_{i+1}^* > 0} (2\text{mult}_p(i) + 1) \right) q^{\sum_{i \geq 1} i \text{mult}_p(i)}.$$

Now observe that for any i , the condition $p_{i+1}^* > 0$, i. e. $\#\{j : p_j \geq i+1\} > 0$, is equivalent to $i < p_1$. Thus the above can be rewritten as

$$F(q) = \sum_p \left(\prod_{i < p_1} (2\text{mult}_p(i) + 1) q^{i \text{mult}_p(i)} \right) q^{p_1 \text{mult}_p(p_1)},$$

or as well

$$F(q) = \sum_n \sum_{p: p_1=n} \left(\prod_{i < n} (2\text{mult}_p(i) + 1) q^{i \text{mult}_p(i)} \right) q^{n \text{mult}_p(n)}.$$

The last expression can be rewritten as

$$\begin{aligned} & \sum_n \sum_{m_1, m_2, \dots, m_{n-1} \geq 0, m_n > 0} ((2m_1 + 1)q^{m_1} \cdots (2m_{n-1} + 1)q^{n-1m_{n-1}}) q^{nm_n} \\ &= \sum_n \left(\sum_{m_1 \geq 0} (2m_1 + 1)q^{m_1} \cdots \sum_{m_{n-1} \geq 0} (2m_{n-1} + 1)(q^{n-1})^{m_{n-1}} \right) \sum_{m_n > 0} (q^n)^{m_n}. \end{aligned}$$

Taking into account that (for $|t| < 1$)

$$\sum_{m \geq 0} (2m + 1)t^m = \frac{1+t}{(1-t)^2}, \quad \sum_{m > 0} t^m = \frac{t}{1-t},$$

we obtain the following

PROPOSITION 4.1. *The generating function for the numbers Pyr_n is given by*

$$F(q) = \sum_{n \geq 1} \left(\prod_{k=1}^{n-1} \frac{1+q^k}{(1-q^k)^2} \right) \frac{q^n}{1-q^n}.$$

THEOREM 4.1. *Let p be a partition of the number n , P a pyramid from $\text{Pyr}(p)$ and $h(P)$ the corresponding diagonal matrix in \mathfrak{gl}_n . Then the pair $(h(P), e(p))$ is good.*

PROOF. First of all, we recall that $e(p)$ is the endomorphism which acts “along the rows of the pyramid” and for this reason it is natural to depict it via horizontal arrows which connect centers of boxes with their right neighbors on the same row. Endomorphisms $E_{i,j}$ map the j th basis vector of \mathbb{F}^n to the i th basis vector, and we represent $E_{i,j}$ by the arrow, which connects the corresponding centers of the boxes of the pyramid P . Then endomorphisms commuting with $e(p)$ are precisely those represented by collections of arrows, which fit with arrows of $e(p)$ into commutative diagrams. Figures 1, 2, 3 represent examples of such commutative diagrams. The loops

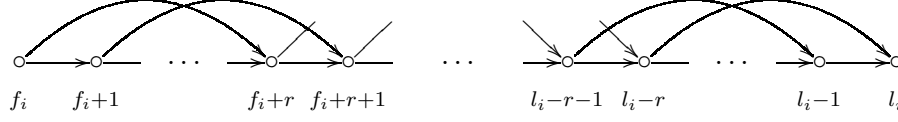


FIGURE 1.

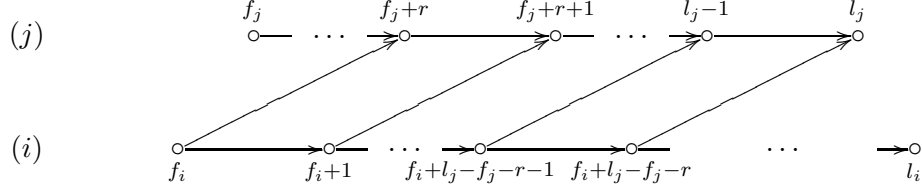


FIGURE 2.

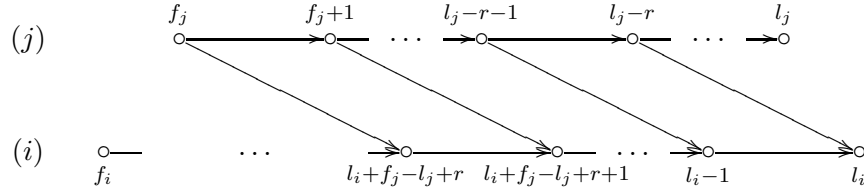


FIGURE 3 .

in these pictures mean identity mappings. Since the end of no arrow in these diagrams is located strictly to the left of its source, all corresponding endomorphisms have non-negative degree with respect to $h(P)$. It is easy to see that when i runs through the set $[1, \dots, k]$, the corresponding endomorphism from figures 1, 2, 3 are linearly independent. The number of the first (resp. 2nd and 3d) type diagrams is $p_1 + p_2 + \dots + p_k$ (resp. $2 \sum_{i=1}^k (i-1)p_i$). It is well known (see [C]) that $n + 2 \sum_{i=1}^k (i-1)p_i = (p_1^*)^2 + \dots + (p_k^*)^2$. All this means that diagrams of types 1, 2, 3, form a basis of the centralizer of the nilpotent $e(p)$ and all elements of this centralizer have non-negative degree with respect to the \mathbb{Z} -grading determined by $h(P)$. \square

THEOREM 4.2. *Let $e(p)$ be the nilpotent element defined by a partition*

$$p = (p_1^{m_1}, p_2^{m_2}, \dots, p_d^{m_d}) = (\underbrace{p_1, \dots, p_1}_{m_1}, \underbrace{p_2, \dots, p_2}_{m_2}, \dots, \underbrace{p_d, \dots, p_d}_{m_d})$$

of the number n . Define $t_i := \sum_{j=1}^{j=i} m_j p_j$ for $1 \leq i \leq d$, so that $t_1 < t_2 < \dots < t_d = n$. Let $P(p)$ be the symmetric pyramid determined by the partition p , let $h(p) := h(P(p))$ be the corresponding diagonal matrix in \mathfrak{gl}_n and let $h = (h_1, h_2, \dots, h_n)$ be a diagonal matrix. Then, the pair $(h(p) + h, e(p))$ is good if and only if the coordinates h_i satisfy the following conditions:

- (1) $h_i - h_j$ are integers,
- (2) $h_1 = h_2 = \dots = h_{t_1}, h_{t_1+1} = \dots = h_{t_2}, \dots, h_{t_{d-1}+1} = \dots = h_{t_d},$
- (3) $|h_{t_1} - h_{t_2}| \leq p_1 - p_2, \dots, |h_{t_{d-1}} - h_{t_d}| \leq p_{d-1} - p_d,$
- (4) $\sum_{i=1}^n h_i = 0.$

Moreover if for each $i \in [1, d-1]$ we set $h_{t_{i-1}} - h_{t_i} = a_i$, where $a_i \in [-p_{i-1} + p_i, p_{i-1} - p_i]$, then the system of linear equations (conditions 2 and 4) has a unique solution.

PROOF. Identify the boxes of the first m_1 rows (resp. the second m_2 rows), \dots , (resp. the last m_d rows) of the symmetric pyramid $P(p)$ with the first t_1 (resp. the second t_2), \dots , (resp. the last t_d) basis vectors of the base space. Then, because the number of boxes in first m_1 rows is equal to p_1 (resp. that in the second m_2 rows is equal to p_2), \dots , (resp. that in the last m_d rows is equal p_d), it follows that for h to be contained in the center of the reductive part of the centralizer, it is necessary and sufficient that the condition 2 be satisfied. Theorem 4.2 gives a transparent description of the basis of the centralizer of $e(p)$. This description together with equalities $E_{i,j} = [E_{i,k}, E_{k,j}]$; $[h(p), e(p)] = 2e(p)$ and condition 2, tell us that non-negativity of h follows from non-negativity of h at the extreme elements of the centralizer, i.e. the elements corresponding to the diagrams in Figures 2 or 3 with $t_i < t_{i+1}$ and $r = 0$. The inequalities 3 are just equivalent to this non-negativity. The last statement of the theorem is clear because the determinant of the corresponding system of linear equations is n . \square

In the notation of Theorem 4.2 put $a_i = 1$ and $a_j = 0$ for $j \neq i$. Then solving the corresponding system of linear equations we obtain $h_1 = \dots = h_{t_{i-1}} = \frac{m_i + \dots + m_d}{n}$ and $h_{t_{i-1}+1} = \dots = h_n = 1 - h_1$. The grading on \mathfrak{sl}_n determined by $h + h(p)$ will not change if we subtract from $h + h(p)$ the scalar matrix $h_1 I_n$. This semisimple element, as is not difficult to see, corresponds to the grading determined by the pyramid obtained from the symmetric pyramid $P(p)$ by shifting to the left by 1 as one whole all rows starting from $t_{i-1} + 1$ up.

Thus, Theorem 4.2 produces a one-to-one correspondence between the pyramids from $\text{Pyr}(p)$ and good gradings for the nilpotent $e(p)$. Consequently Lemma 4.1 gives the number of good pairs of the form $(h + h(p), e(p))$ and the generating function for the number of good gradings of \mathfrak{sl}_n .

Let us recall that a \mathbb{Z} -grading is called even if $\dim \mathfrak{g}_i = 0$ for $i \equiv 1 \pmod{2}$. In the $\mathfrak{sl}(n)$ case a partition p determines an even grading $h(p)$ if and only if all parts p_i of p have the same parity. In terms of the pyramids $P \in \text{Pyr}(p)$ this means that the first coordinates of the centers of all boxes constituting the pyramid P have the same parity.

PROPOSITION 4.2. *Let $e(p)$ be a nilpotent element of \mathfrak{sl}_n determined by a partition p . Then there exists a semisimple element h^e such that the pair $(h^e, e(p))$ is good and h^e determines an even grading.*

PROOF. If the nilpotent $e(p)$ is even, then one can take $h^e = h(p)$, the semisimple element determining the Dynkin grading.

Let $i_1, \dots, i_k \in \{1, \dots, d\}$ be all those natural numbers i for which $p_{i-1} - p_i \equiv 1 \pmod{2}$. Put $a_{i_1} = a_{i_2} = \dots = a_{i_k} = 1$, $a_j = 0$ for $j \neq i_1, \dots, i_k$ (see the notation of Theorem 4.2) and denote by $h(a_{i_1}, \dots, a_{i_k})$ the solution of the corresponding system of equations. Then $h^e := h(a_{i_1}, \dots, a_{i_k}) + h(p)$ will be the required semisimple element. \square

Recall that a *unimodal sequence* of size n is a sequence of natural numbers $h_1 \leq h_2 \leq \dots \leq h_i \geq h_{i+1} \geq \dots \geq h_k$ satisfying $\sum_{j=1}^l h_j = n$.

PROPOSITION 4.3. *There exists a one-to-one correspondence between even good gradings of the simple Lie algebra \mathfrak{sl}_n and unimodal sequences of size n .*

PROOF. Given a unimodal sequence $h = (h_1, \dots, h_k)$ of size n , we construct a pyramid in the following way: the first row of the pyramid will consist of k boxes, with the first coordinates constituting an arithmetic progression with difference 2 and first entry $-k + 1$. The pyramid will consist of $2k - 1$ columns, with columns at even places consisting of 0 boxes and the column at the $(2i - 1)^{\text{th}}$ place consisting of h_i boxes, $i = 1, 2, \dots, k$. This pyramid belongs to the set $\text{Pyr}(h^*)$ and determines an even good grading.

Conversely the sequence of nonzero column heights of a pyramid P determined by an even grading will be an unimodal sequence of size n . It is clear that these two mappings are mutually converse. \square

COROLLARY 4.1. *The generating function for the numbers of even good gradings of the Lie algebras \mathfrak{sl}_n , $n = 1, 2, \dots$, is*

$$U(q) = \sum_{n \geq 1} (-1)^{n+1} q^{\binom{n+1}{2}} \prod_{k \geq 1} \frac{1}{(1 - q^k)^2} \quad (*)$$

PROOF. This follows directly from the previous proposition since according to ([S], Corollary 2.5.3), the generating function for unimodal sequences is $U(q)$. \square

REMARK 4.1. If one looks at the proof of the aforementioned Corollary 2.5.3 from [S], there the generating function is obtained by transforming in a clever way the series

$$\sum_{n \geq 1} \left(\prod_{k=1}^{n-1} \frac{1}{(1 - q^k)^2} \right) \frac{q^n}{1 - q^n},$$

which resembles our generating function $F(q)$ from Proposition 4.1. This makes one wonder whether the latter can be similarly transformed to a more satisfactory form. Now the analog of the second factor of $(*)$ for the series $F(q)$ is more or less obviously the product

$$\prod_{k \geq 1} \frac{1 + q^k}{(1 - q^k)^2},$$

so a natural thing to do is to look at the result of dividing $F(q)$ by this product. The result gives the equality

$$F(q) = \sum_{n \geq 1} \left(\prod_{k=1}^{n-1} \frac{1+q^k}{(1-q^k)^2} \right) \frac{q^n}{1-q^n} = \sum_{n \geq 1} (q^{\frac{3n^2-n}{2}} - q^{\frac{3n^2+n}{2}}) \prod_{k \geq 1} \frac{1+q^k}{(1-q^k)^2}.$$

We are grateful to G. Andrews for providing a proof of this identity.

COROLLARY 4.2. *Let $\mathfrak{p}(a_1, \dots, a_m)$ be a parabolic subalgebra of \mathfrak{sl}_n corresponding to the composition (a_1, \dots, a_m) . Then a Richardson element of $\mathfrak{p}(a_1, \dots, a_m)$ is good for the corresponding even \mathbb{Z} -grading of \mathfrak{sl}_n iff the sequence (a_1, \dots, a_m) is unimodal.*

This result and some necessary conditions for other classical groups were obtained by Lynch [L].

5. Good gradings of \mathfrak{sp}_{2n}

The proofs of the results of this and the next section are similar to those for \mathfrak{sl}_n , and will be omitted.

From now on, given a partition p , we denote by $p_1 > p_2 > \dots$ its distinct non-zero parts and use notation $p = (p_1^{m_1}, \dots, p_s^{m_s})$, where m_i is the multiplicity of p_i in p . A partition is called *symplectic* (resp. *orthogonal*) if m_i is even for odd (resp. even) p_i . Recall that symplectic partitions of $2n$ correspond bijectively to nilpotent orbits in \mathfrak{sp}_{2n} .

Given a symplectic partition p of $2n$, construct a symplectic pyramid $SP(p)$ as follows. It is a centrally symmetric (around $(0,0)$) collection of $2n$ boxes of size 1×1 on the plane, centered at points with integer coordinates (called the coordinates of the corresponding boxes). The 0^{th} row of $SP(p)$ is non-empty iff $m_1 = 2k_1 + 1$ is odd, and in this case the first coordinates of boxes in this row form an arithmetic progression $-p_1 + 1, -p_1 + 3, \dots, p_1 - 1$. The rows from 1^{st} to k^{th} consist of boxes with the first coordinates forming the same arithmetic progression. If the multiplicity m_2 of p_2 is even, then the rows from $k_1^{\text{th}} + 1$ to $(k_1 + \frac{m_2}{2})^{\text{th}}$ consist of boxes with first coordinates forming the arithmetic progression $-p_2 + 1, -p_2 + 3, \dots, p_2 - 1$. However, if m_2 is odd, then the $k_1 + 1^{\text{th}}$ row consists of boxes with first coordinates forming the arithmetic progression $1, 3, \dots, p_2 - 1$ (recall that p_2 must be even if m_2 is odd) and the $m_2 - 1$ subsequent rows consist of boxes with first coordinates forming the arithmetic progression $-p_2 + 1, -p_2 + 3, \dots, p_2 - 1$. All the subsequent parts of p are treated in the same way as p_2 . The rows in the lower half-plane are obtained by the central symmetry.

The nilpotent symplectic endomorphism $e(p)$ of \mathbb{F}^{2n} corresponding to a symplectic partition p is obtained by filling the boxes of $SP(p)$ by vectors $v_1, \dots, v_n, v_{-1}, \dots, v_{-n}$ such that vectors in boxes in the right half-plane ($x \geq 0$ and $y > 0$ if $x = 0$) have positive indices i and those in the centrally symmetric boxes have indices $-i$. Then $e(p)$ maps vectors in each box to its

right neighbor, and to 0 if there is no right neighbor, with the exception of the boxes with coordinates $(-1, -\ell)$ and no right neighbors; the vector in such a box is mapped by $e(p)$ to the vector in the $(1, \ell)$ box (which has no left neighbors).

It is easy to see that the eigenvalue on a vector v_i of the diagonal matrix $h(p) \in \mathfrak{sp}_{2n}$, which defines the corresponding to $e(p)$ Dynkin grading, is equal to the first coordinate of the corresponding box of $SP(p)$.

Recall that, by Theorem 3.1, the dimension of the center of the reductive part of $\mathfrak{sp}_{2n}^{e(p)}$ is equal to $c(p)$, the number of even parts of the partition p having multiplicity 2. If $c(p) = 0$, then, by Corollary 1.1, the only good grading of \mathfrak{sp}_{2n} with good element $e(p)$ is the Dynkin one. Thus, we may assume from now on that $c(p) > 0$.

LEMMA 5.1. *Let $p_1, \dots, p_{c(p)}$ be all distinct even parts of a symplectic partition p , having multiplicity 2. Define diagonal matrices $z(t_1, \dots, t_{c(p)}) \in \mathfrak{sp}_{2n}$, $t_1, \dots, t_{c(p)} \in \mathbb{F}$, whose i^{th} diagonal entry is t_i if the basis vector lies in a box of $SP(p)$ in the (strictly) upper half-plane in a row corresponding to the part p_i , and is $-t_i$ if the basis vector lies in the centrally symmetric box, and all other entries are zero. Then the center of the reductive part of $\mathfrak{sp}_{2n}^{e(p)}$ consists of all these matrices.*

THEOREM 5.1. *In notation of Lemma 5.1, the element $H(p; t_1, \dots, t_{c(p)}) := h(p) + z(t_1, \dots, t_{c(p)})$ defines a good \mathbb{Z} -grading of \mathfrak{sp}_{2n} iff one of the following cases holds:*

- (i) *all parts of p are even and have multiplicity 2, and either all $t_i \in \{-1, 0, 1\}$ or all $t_i \in \{\frac{1}{2}, -\frac{1}{2}\}$;*
- (ii) *not all parts of p are even of multiplicity 2, and all $t_i \in \{-1, 0, 1\}$.*

These \mathbb{Z} -gradings are the same iff the t_i 's differ by signs. Furthermore, these are all good \mathbb{Z} -gradings of \mathfrak{sp}_{2n} for which $e(p)$ is a good element (up to conjugation by the centralizer of $e(p)$ in Sp_{2n}).

COROLLARY 5.1. *A nilpotent element $e(p)$ of \mathfrak{sp}_{2n} is good for at least one even \mathbb{Z} -grading iff it is either even (i.e., its Dynkin grading is even), or it is odd and all even parts of p have multiplicity 2. It is good for at most two even gradings. The element $e(p)$ is good for two even gradings if and only if all parts of p are even and have multiplicity 2; these gradings are given by the elements $H(p; 0, \dots, 0)$ and $H(p; 1, \dots, 1)$, the first one defining the Dynkin grading.*

For each symplectic partition p we construct the set $SPyr(p)$ of symplectic pyramids as follows. If $c(p) = 0$, then $SPyr(p) = \{SP(p)\}$. Let $c(p) > 0$ and let $p_1, \dots, p_{c(p)}$ be all distinct even parts of p which have multiplicity 2. If p contains other parts, to each subset $\{p_{i_1}, \dots, p_{i_s}\}$ we associate a symplectic pyramid obtained from $SP(p)$ by shifting by 1 to the right (resp. left) all boxes from the rows corresponding to p_{i_1}, \dots, p_{i_s} in the (strictly) upper (resp. lower) half-plane. If p does not contain other parts, there is one additional symplectic pyramid $SP_{1/2}(p)$, obtained from $SP(p)$ by shifting all

the rows in the upper half-plane (resp. lower half-plane) by $1/2$ to the right (resp. left). We fill by basis vectors the boxes of a symplectic pyramid and define the corresponding nilpotent element in the same way as for $SP(p)$. Of course, we get the same $e(p)$; the corresponding diagonal matrix $h(SP)$ defining the good gradation, associated with a symplectic pyramid SP , will have the eigenvalue on v_j equal to the first coordinate of the corresponding box, as in the \mathfrak{gl}_n case.

The characteristic of this good \mathbb{Z} -grading is obtained as follows. If SP is different from $SP_{1/2}(p)$, we denote by h_i , $0 \leq i \leq p_1 + 1$, the number of boxes of SP whose first coordinates are $p_1 + 1 - i$. For each $h_j > 0$, $0 \leq j \leq p_1$ construct a sequence, which begins with $h_j - 1$ zeros followed by 2 if $h_{j+1} = 0$ and by 1 if $h_{j+1} > 0$. The number $h_{p_1+1} = 2\ell$ is even (since this is the number of boxes in the 0^{th} row). Concatenating the obtained sequences and adding ℓ zeros, we obtain a sequence of n numbers equal to 0, 1 or 2, which defines the characteristic by assigning these integers to the corresponding simple roots depicted by the Dynkin diagram $\circ - \circ - \dots - \circ - \circ \Leftarrow \circ$.

In the case $SP = SP_{1/2}(p)$, define h_i as the number of boxes in SP , whose first coordinate is $p_1 - i + 3/2$. We construct the sequences for $0 \leq i \leq p_1$ as before. The number $h_{p_1+1} = 2\ell$ is even, and we define the $p_1 + 1^{\text{th}}$ sequence consisting of $2\ell - 1$ zeros and one 1. Concatenating these $p_1 + 2$ sequences we obtain the characteristic of $SP_{1/2}(p)$ (consisting of zeros and 1's).

THEOREM 5.2. *Let $\mathfrak{p}(a_1, \dots, a_t; q)$ be a parabolic subalgebra of \mathfrak{sp}_{2n} corresponding to the pair $(a_1, \dots, a_t; q)$. Then a Richardson element of $\mathfrak{p}(a_1, \dots, a_t; q)$ is good for the corresponding even grading of \mathfrak{sp}_n iff the following two properties hold:*

- (i) $0 < a_1 \leq a_2 \leq \dots \leq a_t$,
- (ii) if $q > 0$, then $a_t \leq q$ and the multiplicities of all odd a_j are ≤ 1 .

6. Good gradings of \mathfrak{so}_n

Given an orthogonal partition p of N , construct an orthogonal pyramid $OP(p)$ as follows. It is a centrally symmetric (around $(0, 0)$) collection of N boxes of size 1×1 on the plane, centered at points with integer coordinates (called the coordinates of the corresponding boxes). First, consider the case $N = 2n$ is even. In this case the 0^{th} row is empty. If p_1 has multiplicity m_1 , then the rows from the 1^{st} to the $[\frac{m_1}{2}]^{\text{th}}$ consist of boxes with the first coordinates forming the arithmetic progression $-p_1 + 1, -p_1 + 3, \dots, p_1 - 1$. In the case when $m_1 = 2k_1 + 1$ is odd, we add the $k_1 + 1^{\text{st}}$ row as follows. Let p_i be the greatest part among $p_2 > p_3 > \dots$ with odd multiplicity; then the $k_1 + 1^{\text{st}}$ row consists of boxes whose first coordinates form the arithmetic progression $-p_i + 1, -p_i + 3, \dots, 0, 2, \dots, p_1 - 1$ (recall that both p_1 and p_i are odd). After that we remove the parts p_1 and p_i from the partition p and

continue building up the pyramid for the remaining parts. The rows in the lower half-plane are obtained by the central symmetry.

In the case when N is odd, $OP(p)$ always contains the box with coordinates $(0, 0)$. If the multiplicity of p_1 is even, then the 0^{th} row contains no other boxes, and all the other rows of the pyramid $OP(p)$ are constructed in the same way as for N even. If the multiplicity of p_1 is odd, then p_1 is odd, and the 0^{th} row consists of boxes whose first coordinates form the arithmetic progression $-p_1 + 1, -p_1 + 3, \dots, p_1 - 1$. The partition $p' = (p_2^{m_2}, p_3^{m_3}, \dots)$ is a partition of an even integer, and the non-zero rows of $OP(p)$ simply coincide with the corresponding rows of $OP(p')$.

We fill the boxes of $OP(p)$ by basis vectors in the same way as for \mathfrak{sp}_N , except that in the case of \mathfrak{so}_N , N odd, the vector v_0 is put in the 0^{th} box. The nilpotent $e(p)$, associated to the orthogonal partition p , is defined as the endomorphism which maps each basis vector to its right neighbor, and to 0 if there is no right neighbor, with the following exceptions. If N is odd and the 0^{th} row consists only of the box $(0, 0)$, the vector in the box with coordinates $(-2, -\ell)$ and no right neighbor is mapped to v_0 , and v_0 is mapped to the vector in the box $(2, \ell)$ (which has no left neighbors). If N is even, then $e(p)$ is the sum of the operator defined by horizontal maps as above and the sum over all pairs of unequal parts p_i, p_j with odd multiplicities of maps which send the vector from the box of the row corresponding to the pair (i, j) with coordinates $(0, -d)$ to that with coordinates $(2, d)$ and all other basis vectors to 0.

LEMMA 6.1. *Let $p_1, \dots, p_{c(p)}$ be all odd parts of an orthogonal partition p , having multiplicity 2. Define diagonal matrices $z(t_1, \dots, t_{c(p)}) \in \mathfrak{so}_N$, $t_1, \dots, t_{c(p)} \in \mathbb{F}$, whose i^{th} diagonal entry is t_i if the basis vector lies in a box of $OP(p)$ in the (strictly) upper half-plane corresponding to the part p_i , and is $-t_i$ if the basis vector lies in the centrally symmetric box, all other entries being zero. Then the center of the reductive part of $\mathfrak{so}_N^{e(p)}$ consists of all these matrices.*

Define $h(p) \in \mathfrak{so}_N$, using the pyramid $OP(p)$ in the same way as for \mathfrak{sp}_N .

THEOREM 6.1. *In notation of Lemma 6.1, the element $H(p; t_1, \dots, t_{c(p)}) := h(p) + z(t_1, \dots, t_{c(p)})$ defines a good \mathbb{Z} -grading of \mathfrak{so}_{2n+1} iff one of the following cases holds:*

- (i) 1 does not have multiplicity 2 in p and all $t_i \in \{-1, 0, 1\}$;
- (ii) 1 has multiplicity 2 in p and $p_{c(p)-1}$ is the smallest part of p not equal to 1, $t_i \in \{-1, 0, 1\}$ for $1 \leq i \leq c(p) - 1$, $t_{c(p)} \in \mathbb{Z}$, and $|t_{c(p)-1} - t_{c(p)}| \leq p_{c(p)-1} - 1$, $|t_{c(p)-1} + t_{c(p)}| \leq p_{c(p)-1} - 1$;
- (iii) 1 has multiplicity 2 in p and $p_{c(p)-1} \neq q$, where q is the smallest part of p greater than 1, $t_i \in \{-1, 0, 1\}$ for $1 \leq i \leq c(p) - 1$, $t_{c(p)} \in \mathbb{Z}$ and $|t_{c(p)}| \leq q - 1$.

These gradings are the same iff the t_i differ by signs. Furthermore, these are all good \mathbb{Z} -gradings of \mathfrak{so}_{2n+1} for which $e(p)$ is a good element (up to conjugation by the centralizer of $e(p)$ in SO_{2n+1}).

THEOREM 6.2. *In notation of Lemma 6.1, let p be an orthogonal partition of $2n$ and let $C(p) = \{p_1 > \dots > p_{c(p)}\}$ be all distinct odd parts of p with multiplicity 2. The element $H(p; t_1, \dots, t_{c(p)}) := h(p) + z(t_1, \dots, t_{c(p)})$ defines a good \mathbb{Z} -grading of \mathfrak{so}_{2n} with nilpotent $e(p)$ iff one of the following cases holds:*

- (i) $1 \notin C(p)$, there is a part p_i of p such that $p_i \notin C(p)$ and all $t_i \in \{-1, 0, 1\}$;
- (ii) $1 \in C(p)$, there is a part p_i of p such that $p_i \notin C(p)$ and $p_{c(p)-1}$ is the smallest part of p not equal to 1, $t_i \in \{-1, 0, 1\}$ for $1 \leq i \leq c(p) - 1$, $t_{c(p)} \in \mathbb{Z}$, and $|t_{c(p)-1} - t_{c(p)}| \leq p_{c(p)-1} - 1$, $|t_{c(p)-1} + t_{c(p)}| \leq p_{c(p)-1} - 1$;
- (iii) $1 \in C(p)$, there is a part p_i of p such that $p_i \notin C(p)$ and $p_{c(p)-1} \neq q$, the smallest part of p greater than 1, $t_i \in \{-1, 0, 1\}$ for $1 \leq i \leq c(p) - 1$, $t_{c(p)} \in \mathbb{Z}$, and $|t_{c(p)}| \leq q - 1$;
- (iv) $1 \notin C(p)$, all parts p_i of p lie in $C(p)$, $2t_i, t_i + t_j, t_i - t_j \in \mathbb{Z}$, $-2 \leq 2t_i \leq 2$ for $1 \leq i < j \leq c(p)$;
- (v) $1 \in C(p)$, all parts p_i of p lie in $C(p)$, $2t_i, t_i + t_j, t_i - t_j \in \mathbb{Z}$, for $1 \leq i < j \leq c(p) - 1$, $-2 \leq 2t_i \leq 2$ for $1 \leq i < j \leq c(p) - 1$, $|t_{c(p)-1} - t_{c(p)}| \leq p_{c(p)-1} - 1$, $|t_{c(p)-1} + t_{c(p)}| \leq p_{c(p)-1} - 1$.

These gradings are the same iff the t_i differ by signs. Furthermore, these are all good \mathbb{Z} -gradings of \mathfrak{so}_{2n} for which $e(p)$ is a good element (up to conjugation by the centralizer of $e(p)$ in O_{2n}).

Next, for each orthogonal partition p of N we construct the set $OPyr(p)$ of orthogonal pyramids. If $c(p) = 0$, then $OPyr(p) = \{OP(p)\}$. Assume now that $c(p) > 0$, and let $C(p) = \{p_1 > \dots > p_{c(p)}\}$ be all distinct odd parts of p with multiplicity 2.

First consider the case of N odd. If $1 \notin C(p)$, to each subset $\{p_{i_1}, \dots, p_{i_s}\}$ of $\{p_1, \dots, p_{c(p)}\}$ we associate an orthogonal pyramid obtained from $OP(p)$ by shifting by 1 to the right (resp. left) all boxes from the rows corresponding to p_{i_1}, \dots, p_{i_s} , in the strictly upper (resp. lower) half-plane. If $1 \in C(p)$, i.e., $p_{c(p)} = 1$, let q be the smallest part of the partition p , which is greater than 1. If q is not an odd part with multiplicity 2, then for each pair $(\{p_{i_1}, \dots, p_{i_s}\}, t_{c(p)})$, where $\{p_{i_1}, \dots, p_{i_s}\} \subset \{p_1, \dots, p_{c(p)-1}\}$ and $0 \leq t_{c(p)} \leq q - 1$, $t_{c(p)} \in \mathbb{Z}$, we construct an orthogonal pyramid obtained from $OP(p)$ by shifting by 1 to the right (resp. left) all boxes from the rows corresponding to the parts p_{i_1}, \dots, p_{i_s} , and the box corresponding to part 1, by $t_{c(p)}$ to the right (resp. left) in the strictly upper (resp. lower) half plane. Under these assumptions we obtain $2^{c(p)-1}q$ orthogonal pyramids, which form the set $OPyr(p)$. If, finally, $q = p_{c(p)-1}$ and $p_{c(p)} = 1$, then for each pair $(\{p_{i_1}, \dots, p_{i_s}\}, t_{c(p)})$ (resp. $(\{p_{i_1}, \dots, p_{i_s}, p_{c(p)-1}\}, t_{c(p)})$), where

Let now N be even. If p has parts which are not in $C(p)$, then $OPyr(p)$ is constructed in the same way as for N odd. Finally, suppose all parts of p are odd of multiplicity 2. If $1 \notin C(p)$, then $OPyr(p)$ is constructed in the same way as $SPyr(p)$. If $1 \in C(p)$, then $OPyr(p)$ consists of the pyramids constructed as for N odd and the set of pyramids $OP_{1/2, p_{c(p)-1}}(p)$ obtained by a shift by $1/2$ to the right (resp. left) of all rows corresponding to the parts $p_1, \dots, p_{c(p)-1}$ and the box from the row, corresponding to 1, by t to the right (resp. left) in the upper (resp. lower) half-plane, where $t \in \frac{1}{2} + \mathbb{Z}$, $\frac{1}{2} \leq t \leq p_{c(p)-1} - \frac{3}{2}$.

In the case $OP_{1/2, p_c(p)-1}(p)$, define h_i as the number of boxes in SP , whose first coordinate is $p_1 - i + 3/2$. We construct the sequences for $0 \leq i \leq p_1$ as before. The number $h_{p_1+1} = 2\ell$ is even, and we define the $p_1 + 1^{\text{th}}$ sequence consisting of $2\ell - 1$ zeros and one 1. Concatenating these $p_1 + 2$ sequences we obtain the characteristic of $OP_{1/2, p_c(p)-1}(p)$ (consisting of zeros and 1's).

LEMMA 6.2. *The nilpotent $e(p)$ of \mathfrak{so}_N , N odd, for a partition p of N is good for an even grading iff all parts p_i are odd.*

By Lemma 6.2 and Theorem 6.1 the nilpotent $e(p)$ can be good for even non-Dynkin gradings if and only if $1 \in C(p)$ and all parts p_i of p are odd.

We now apply the above algorithm for constructing the characteristic of the grading for an orthogonal pyramid from $OPyr(p)$ to obtain

THEOREM 6.3. *Let $p = \mathfrak{p}(a_1, \dots, a_t; q)$ be a parabolic subalgebra of \mathfrak{so}_{2n+1} corresponding to the pair $(a_1, \dots, a_t; q)$ (recall that in this case q is odd). Then a Richardson element of $\mathfrak{p}(a_1, \dots, a_t; q)$ is good for the corresponding even grading of \mathfrak{so}_{2n+1} iff one of the following properties holds:*

- (i) $0 < a_1 \leq a_2 \leq \dots \leq a_t \leq q$;
- (ii) p has the form $\mathfrak{p}(a_1, \dots, a_{t-2}, a_{t-1}^{m_{t-1}-s}, a_t, a_{t-1}^s; q)$, where $0 < a_1 \leq \dots \leq a_{t-2} < a_{t-1} = q < a_t = q + 1$ and $0 \leq s \leq m_{t-1}$;
- (iii) $0 < a_1 \leq a_2 \leq \dots \leq a_{t-1} < q < a_t = q + 1$.

LEMMA 6.3. *The nilpotent $e(p)$ of \mathfrak{so}_{2n} for a partition p of $2n$ is good for an even grading iff one of the two following properties holds:*

- (i) p is an even partition;
- (ii) if p is not even then all odd parts p_i of p have multiplicity $m_i = 2$.

By Lemma 6.3 and Theorem 6.2 the nilpotent $e(p)$ can be good for an even non-Dynkin grading iff $1 \in C(p)$ and all parts p_i of p are odd. We now apply the above algorithm for constructing the characteristic of the grading for an orthogonal pyramid from $OPyr(p)$ to obtain

THEOREM 6.4. *Let $\mathfrak{p} = \mathfrak{p}(a_1, \dots, a_t; q)$ be a parabolic subalgebra of \mathfrak{so}_{2n} corresponding to the pair $(a_1, \dots, a_t; q)$ (recall that in this case q is even $\neq 2$). Then a Richardson element of $\mathfrak{p}(a_1, \dots, a_t; q)$ is good for the corresponding even grading of \mathfrak{so}_{2n} iff one of the following properties holds:*

- (i) $n = 2s + 1, q = 0$ and composition (a_1, \dots, a_t) has one of the following forms:
 - $(1^{2(s-i)}, 2^i, 1)$, where $i = 0, 1, \dots, s$;
 - $(1^{2(s-i)+1}, 2^{i-l}, 3, 2^{l-2}, 1)$, where $i = 2, \dots, s, l = 2, \dots, i$;
 - $(2^{s-(i+1)}, 3, 2^i)$, where $i = 1, \dots, s - 1$;
- (ii) $n = 2s + 2, q = 0$ and composition $(a_1^{m_1}, \dots, a_t^{m_t})$ has one of the following forms:
 - $(1^{2(s-i)+1}, 2^i, 1)$, where $i = 0, 1, \dots, s$;
 - $(1^{2(s-i)}, 2^{i-l}, 3, 2^{l-1}, 1)$, where $i = 1, \dots, s, l = 1, \dots, i$;
 - $(1, 2^{(i-1)}, 3, 2^{s-i})$, where $i = 1, \dots, s - 1$;
- (iii) $q = 0, 0 < a_1 \leq a_2 \leq \dots \leq a_t$, and multiplicities of odd a_j are at most 1;
- (iv) $q = 0, 0 < a_1 \leq a_2 \leq \dots \leq a_{t-1}, a_t = a_{t-1} - 1 > 0$, where a_{t-1} is odd, $a_{t-1} \geq 5, m_t > 0$, and multiplicities of odd a_j are at most 1;
- (v) $q > 0, 0 < a_1 \leq a_2 \leq \dots \leq a_t \leq q$;
- (vi) \mathfrak{p} has the form $\mathfrak{p}(a_1, \dots, a_{t-2}, a_{t-1}^{m_{t-1}-s}, a_t, a_{t-1}^s; q)$, where $0 < a_1 \leq a_2 \leq \dots \leq a_{t-2} < a_{t-1} = q < a_t = q + 1, m_{t-1} > 0$, and $0 \leq s \leq m_{t-1}$;
- (vii) $0 \leq a_1 \leq a_2 \leq \dots \leq a_{t-1} < q < a_t = q + 1$.

7. Good gradings of exceptional simple Lie algebras

If \mathfrak{g} is an exceptional simple Lie algebra G_2 or F_4 , then, due to the tables of [E], the reductive part of \mathfrak{g}^e is semisimple (or zero) for any nilpotent element e of \mathfrak{g} . Hence, by Corollary 1.1, we have

THEOREM 7.1. *All good \mathbb{Z} -gradings of G_2 and F_4 are the Dynkin ones.*

Due to the tables of [E] for the exceptional simple Lie algebras E_6 , E_7 and E_8 the number of nilpotent orbits for which the reductive part of the centralizer is not semisimple (and not zero) is 10, 6 and 7, respectively (out of 21, 45 and 70, respectively). In notation of [C] they are as follows:

$$\begin{aligned} E_6 &: D_5, D_5(a_1), A_4 + A_1, D_4(a_1), A_4, A_3 + A_1, A_3, A_2 + 2A_1, A_2 + A_1, 2A_1; \\ E_7 &: E_6(a_1), D_5(a_1), A_4 + A_1, A_4, A_3 + A_2, A_2 + A_1; \\ E_8 &: D_7(a_1), E_6(a_1) + A_1, D_7(a_2), D_5 + A_2, A_4 + 2A_1, A_4 + A_1, A_3 + A_2. \end{aligned}$$

We use Theorem 1.1 in order to describe all good \mathbb{Z} -gradings for which the above listed nilpotent elements are good. The answer is given in Tables E_6 , E_7 and E_8 , where we list all nilpotent orbits which admit non-Dynkin good gradings. (The Dynkin gradings are described in [D], [E], [C].)

THEOREM 7.2. *All good non-Dynkin \mathbb{Z} -gradings of E_6 , E_7 and E_8 are described in the second column of Tables E_6 , E_7 and E_8 , respectively, where we list all good non-Dynkin \mathbb{Z} -gradings for which the nilpotent of the first column is good. In Table E_6 the characteristics of good non-Dynkin \mathbb{Z} -gradings are listed up to the symmetry of the Dynkin diagram.*

Note that among the 10 nilpotent orbits of E_6 there is exactly one for which the reductive part is not semisimple, but the Dynkin grading is the only good one (and for E_7 and E_8 there are 2 and 5 such nilpotents, respectively). This is the nilpotent of type $A_2 + A_1$. Pick one such nilpotent e , and let $h(e)$ be the semisimple element which defines the corresponding Dynkin grading of E_6 . The reductive part of its stabilizer is a direct sum of A_2 and a 1-dimensional torus T_1 [E]. Among the weights of $h(e) + T_1$ on E_6^e there are weights $1 \pm t$ and $1 + 3t$. Hence $t \in \mathbb{Z}$ and the condition that $(E_6^e)_j = 0$ for $j < 0$ gives the inequalities $-\frac{1}{3} \leq t \leq \frac{1}{3}$. Hence $t = 0$. Similar argument is used to describe all good \mathbb{Z} -gradings for all other nilpotents as good elements.

Let us give one more example—the nilpotent e of type A_4 in $\mathfrak{g} = E_6$. It is easy to see that for the corresponding Dynkin \mathbb{Z} -grading of E_6 one has:

$$\dim \mathfrak{g}_0 = 18, \dim \mathfrak{g}_2 = 14, \dim \mathfrak{g}_4 = 9, \dim \mathfrak{g}_6 = 6, \dim \mathfrak{g}_8 = 1.$$

Hence, by (0.2), we have:

$$\dim \mathfrak{g}_0^e = 4, \dim \mathfrak{g}_2^e = 5, \dim \mathfrak{g}_4^e = 3, \dim \mathfrak{g}_6^e = 5, \dim \mathfrak{g}_8^e = 1.$$

The reductive part of \mathfrak{g}^e is $\mathfrak{g}_0^e \simeq A_1 \oplus T_1$, where $T_1 = \{t | t \in \mathbb{F}\}$ is the 1-dimensional torus. From the explicit description of the basis of \mathfrak{g}^e , which can

TABLE E_6 .

D_5	2 0 2 0 2 2	2 1 1 1 1 2 2 0 2 0 2 2 2 1 1 1 2 2 2 0 2 1 0
$D_5(a_1)$	1 1 0 1 1 2	2 0 0 2 0 2
$A_4 + A_1$	1 1 0 1 1 1	2 0 2 0 0 0
$D_4(a_1)$	0 0 2 0 0 0	1 0 1 1 0 1 1 0 1 1 0 0 2 0 0 2 0 0
A_4	2 0 0 0 2 2	2 1 0 1 0 2 0 0 2 2 1 0
$A_3 + A_1$	0 1 0 1 0 1	1 1 0 0 1 2 0 1 0 1 1 0
A_3	1 0 0 0 1 2	2 0 0 0 0 2 1 0 0 0 2 1 2 2 0 0 0 0
$A_2 + 2A_1$	0 1 0 1 0 0	0 2 0 0 0 0
$2A_1$	1 0 0 0 1 0	2 0 0 0 0 0

TABLE E_7 .

$E_6(a_1)$	0 2 0 2 0 2 0	1 1 1 1 0 2 2 0 2 0 0 2 1 2 2 1 1 0 1 1 2 2 0 0 2 0 2 2
$D_5(a_1)$	0 1 0 1 0 2 0	2 0 0 0 0 2 2
$A_4 + A_1$	0 1 0 1 0 1 0	1 0 1 0 0 1 2 0 2 0 0 0 1 0 2 0 1 0 1 0 2 0 0 0 2 0 0 0
A_4	0 2 0 0 0 2 0	2 1 0 0 0 1 1

TABLE E_8 .

$D_7(a_1)$	2	0	0	2	0	0	2	1	1	1	0	0	1	1
					0							1		
$D_7(a_2)$	1	0	1	0	1	0	1	0	2	0	0	0	2	0
					0							0		

be obtained by using the explicit form of e in [C], [D], [E], and the computer program GAP (see www-gap.dcs.st-and.ac.uk/~gap/), we find the weights of $h(e) + T_1$ on \mathfrak{g}_j^e ($j \in 2\mathbb{Z}_+$): $2 \pm 3t$ and $2; 4 \pm 6t$ and $4; 6 \pm 3t; 6; 8$. The conditions of integrability and non-negativity of a good \mathbb{Z} -grading give the following restrictions:

$$3t \in \mathbb{Z}, \quad -2 \leq 3t \leq 2.$$

Hence $t = 0, \frac{1}{3}, \frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}$ give all good \mathbb{Z} -gradings with e a good element. Of course, $t = 0$ gives the Dynkin grading, and it is easy to see that $t = \frac{1}{3}$ and $\frac{2}{3}$ give the good gradings in the line A_4 of Table E_6 . The values $t = -\frac{1}{3}$ and $-\frac{2}{3}$ give the good \mathbb{Z} -gradings obtained by the symmetry of the Dynkin diagram.

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